Introduction to the Standard Model
William and Mary PHYS 771 Spring 2014

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Class information, including syllabus and homework assignments can be found at
`http://ntc0.lbl.gov/~walkloud/wm/courses/PHYS_771/`
or
`http://cyclades.physics.wm.edu/~walkloud/wm/PHYS_771/`

Homework Assignment 4: due Wednesday 30 April

1. **[20 pts.]** We discussed the QCD beta function, which at one-loop gives the strong coupling

$$\begin{aligned} \alpha_S(\mu) &= \frac{\alpha_S(Q_0)}{1 + \frac{\alpha_S(Q_0)}{4\pi}\beta_0 \ln\left(\frac{\mu^2}{Q_0^2}\right)} \\ &= \frac{1}{\frac{\beta_0}{4\pi}\ln\left(\frac{\mu^2}{\Lambda_{QCD}^2}\right)} \end{aligned} \tag{1}$$

where

$$\beta_0(N_c = 3) = \frac{33 - 2n_f}{3}\,, \qquad \text{and} \qquad \Lambda_{QCD}^2 = Q_0^2 \exp\left\{-\frac{4\pi}{\alpha_S(Q_0)\beta_0}\right\}\,. \tag{2}$$

In these expressions, n_f is the number of "active" quark flavors, meaning quarks with $m_q < \mu$. Even for massless quarks, QCD dynamically generates an energy scale, Λ_{QCD}, which is known as "Dimensional Transmutation". This scale is simply defined at a given order in perturbation theory as the scale where the coupling diverges. Before reaching this scale, of course the theory becomes non-perturbative, and so this scale provides only a qualitative understanding of the "scale of QCD". Qualitatively, hadrons comprised of u, d and s quarks, whose mass is not protected by chiral symmetry (the pions etc.), have a mass proportional to Λ_{QCD} with corrections from the light quark masses

$$m_H = c_H \Lambda_{QCD} + \mathcal{O}(m_q) \tag{3}$$

which is why the proton is $m_p \sim 1$ GeV.

(a) Fix α_S at the Z-pole. Using the one-loop running, what is $\Lambda_{QCD} =$?
To answer this question, you need to start with the first line of Eq. (1), and run the scale μ down through the heavy quark thresholds. At $\mu = m_q$, you match the the coupling above and below m_q where the running uses different number of "active" flavors above and below the scale $\mu = m_q$. e.g. below M_Z but above m_b, you have 5 active flavors while for $\mu < m_b$, there are only 4 active flavors. Perform this matching and running until you find a scale at which the coupling diverges.

[10 pts.] *Solution:*
With the benefit of hindsight, we know the value of $m_s < \Lambda_{QCD} < m_c$, simplifying the solution. (The full solution is not much more involved). We can simply bootstrap backwards

$$\begin{aligned}
\Lambda_{QCD} &= m_c \exp\left(-\frac{2\pi}{\alpha_S(m_c)\beta_0(n_f=3)}\right)\,, \\
\alpha_S(m_c) &= \frac{\alpha_S(m_b)}{1+\frac{\alpha_S(m_b)}{4\pi}\beta_0(n_f=4)\ln\left(\frac{m_c^2}{m_b^2}\right)}\,, \\
\alpha_S(m_b) &= \frac{\alpha_S(M_Z)}{1+\frac{\alpha_S(M_Z)}{4\pi}\beta_0(n_f=5)\ln\left(\frac{m_b^2}{M_Z^2}\right)}\,,
\end{aligned}$$

which requires the following input (taken from the PDG online)[1]

$$\begin{aligned}
\alpha_S(M_Z) &= 0.1185(06)\,, \qquad & m_c &= 1.275(25)\text{ GeV}\,, \\
M_Z &= 91.1876(21)\text{ GeV}\,, \qquad & m_b &= 4.18(3)\text{ GeV}\,.
\end{aligned}$$

These values provide the determination

$$\alpha_S(m_b) \simeq 0.214\,, \qquad \alpha_S(m_c) \simeq 0.322\,, \qquad \Lambda_{QCD} \simeq 146\text{ MeV}\,.$$

(b) What would be the value Λ_{QCD} if $m_b > M_Z$?
[5 pts.] *Solution:*
If $m_b > M_Z$, we would have a similar solution as above, except we would determine $\alpha_S(m_c)$ by running the coupling down from M_Z without the intermediate determination of $\alpha_S(m_b)$. Thus, we have

$$\alpha_S(m_c) \simeq 0.360\,, \qquad \Lambda_{QCD} \simeq 184\text{ MeV}\,.$$

(c) What would be the value Λ_{QCD} if $m_b = 50$ GeV?
[3 pts.] *Solution:*
For $m_b = 50$ GeV, we again have a similar solution to (a) with just a different value of m_b used in the determination:

$$\alpha_S(m_b) \simeq 0.130\,, \qquad \alpha_S(m_c) \simeq 0.352\,, \qquad \Lambda_{QCD} \simeq 176\text{ MeV}\,.$$

[1] We take the $\overline{\text{MS}}$ masses determined at $\alpha_S(\mu = m_q)$ for the quark masses. This is consistent with the $\overline{\text{MS}}$ scheme used in the determination of $\alpha_S(\mu)$. It is important to take consistent renormalization schemes. In this case, it is less of a concern as Λ_{QCD} is a phenomenological quantity which is not observable. But to compare this value with other QCD calculations, it is important to pick a scheme and clearly specify it. Since we are working to only 1-loop, many choices are consistent. The choice of the PDG values of the quark masses, which come from 4-loop QCD running provide a convenient choice consistent to the order we are working.

(d) if the b-quark mass were to increase, would the mass of the proton increase or decrease? (explain)

[**2 pts.**] *Solution:*

The point of this entire problem was to build intuition for how the dynamics of QCD (the running coupling constant) has an influence of the values low-energy hadronic physics. The gluons and quarks give opposite signs to the running of the coupling, and so for fixed $N_c = 3$, the more "active quark flavors" that participate in the loops, the slower the coupling runs. We see as we increase the mass of the b-quark, the value of Λ_{QCD} increases, because there is a longer window in energy for which the QCD coupling runs with less active flavors, and thus changes more rapidly, in this case, increasing faster as we lower the scale μ. Therefore, increasing the mass of the b-quark leads to an increase in the mass of the proton.

2. [**55 pts.**] Cottingham's Formula and the electron electromagnetic self-energy. In class, we discussed the Cottingham Formula and the nucleon electromagnetic self-energy. Here, we will use it to determine the electron self-energy. Cottingham's Formula is

$$\delta M^\gamma = \frac{i}{2M}\frac{\alpha_{f.s.}}{(2\pi)^3}\int_R d^4q \frac{g^{\mu\nu}T_{\mu\nu}(q^0, -q^2)}{q^2 + i\epsilon} \tag{4a}$$

$$= \frac{\alpha_{f.s.}}{8M\pi^2}\int_R dQ^2 \int_{-Q}^{+Q} d\nu \frac{\sqrt{Q^2-\nu^2}}{Q^2} T^\mu_\mu(i\nu, Q^2) \tag{4b}$$

where the subscript R reminds us the integral must be renormalized.

(a) Derive Eq. (4b) from Eq. (4a).

[**5 pts.**] *Solution:* see attached notes

(b) Starting from

$$T_{\mu\nu}(q^0, -q^2) = \frac{i}{2}\sum_\sigma \int d^4\xi e^{iq\cdot\xi}\langle p,\sigma|T\{J_\mu(\xi), J_\nu(0)\}|p,\sigma\rangle \tag{5}$$

this forward Compton Amplitude is crossing symmetric,

$$T_{\nu\mu}(-q^0, q^2) = T_{\mu\nu}(q^0, q^2): \tag{6}$$

Show this to be true.

[**5 pts.**] *Solution:* see attached notes

(c) Use this crossing symmetry to show the scalar functions satisfy

$$T_i(-q^0, -q^2) = T_i(q^0, -q^2) \tag{7}$$

where the $T_i(q^0, -q^2)$ are defined for example below in Eq. (8).

[**0 pts.**] *Solution:* added late, will not count

(d) At leading order in QED, what is the electron forward Compton Scattering Amplitude?

[**5 pts.**] *Solution:* see attached notes

i. using the parameterization,

$$T_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) T_1(q^0, -q^2) + \frac{1}{M^2}\left(p_\mu - q_\mu \frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu \frac{p\cdot q}{q^2}\right) T_2(q^0, -q^2) \tag{8}$$

what are the scalar functions $T_i(q^0, -q^2) =$?
[10 pts.] *Solution:* see attached notes

ii. using the parameterization,

$$T_{\mu\nu} = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) q^2 t_1(q^0, -q^2) - \frac{1}{M^2}\left(p_\mu p_\nu - \frac{p\cdot q}{q^2}(p_\mu q_\nu + p_\nu q_\mu) + \frac{(p\cdot q)^2}{q^2} g_{\mu\nu}\right) q^2 t_2(q^0, -q^2) \tag{9}$$

A. what is the relation between t_i and T_i?
[5 pts.] *Solution:* see attached notes

B. what are the scalar functions $t_i(q^0, -q^2) =$?
[5 pts.] *Solution:* see attached notes

(e) Using your determination of the forward Compton Amplitude, evaluate the self-energy in Eq. (4b). To perform this evaluation, use Pauli-Villars with a Q^2 cut-off. Recall, Pauli-Villars replaces the photon propagator with the difference between the photon and a heavy photon. In our Eq. (4b), this amounts to

$$\frac{1}{Q^2} \to \frac{1}{Q^2} - \frac{1}{Q^2 + \Lambda^2} \tag{10}$$

and to make the Q^2 integral finite, we can put in a UV cutoff and add a counterterm, such that our mass self-energy correction becomes

$$\delta M^\gamma = \lim_{Q_{UV}\to\infty}\left[\frac{\alpha_{f.s.}}{8M\pi^2}\int_0^{Q^2_{UV}} dQ^2 \int_{-Q}^{+Q} d\nu \sqrt{Q^2 - \nu^2}\, T^\mu_\mu(i\nu, Q^2)\left[\frac{1}{Q^2} - \frac{1}{Q^2+\Lambda^2}\right] + \delta M(\Lambda)\right] \tag{11}$$

where $\delta M(\Lambda)$ is the counterterm needed to render the answer independent of Λ.

i. Evaluate the integral and take the large-Λ limit. What is the resulting expression for the self-energy correction including the finite and logarithmic terms?
[10 pts.] *Solution:* see attached notes

ii. Use the ideas of renormalization to determine the counterterm (demand the entire answer be independent of Λ, usually done by taking $\partial/\partial \ln(\Lambda^2)\delta M^\gamma = 0$).
[5 pts.] *Solution:* see attached notes

iii. How does your answer compare with the answer using dimensional regularization? (or compare with the answer in the literature/books)
[5 pts.] *Solution:* see attached notes

Cottingham Formula

$$\delta M^{\gamma} = \frac{i}{2M} \frac{\alpha_{f.s.}}{(2\pi)^3} \int_R d^4q \, \frac{g^{\mu\nu} T_{\mu\nu}(q^0, -q^2)}{q^2 + i\varepsilon} \qquad 4a$$

$$= \frac{\alpha_{f.s.}}{8M\pi^2} \int_R dQ^2 \int_{-Q}^{Q} d\nu \, \frac{\sqrt{Q^2-\nu^2}}{Q^2} \, T^{\mu}_{\mu}(i\nu, Q^2) \qquad 4b$$

a) Show (4b) follows from (4a)

$$\delta M^{\gamma} = \frac{i}{2M} \frac{\alpha_{f.s}}{(2\pi)^3} \int_R dq^0 \int d^3q \, \frac{T_{\mu}^{\mu}(q^0, -q^2)}{q^{0\,2} - \vec{q}^{\,2} + i\varepsilon}$$

$$q^0 \to i\nu$$

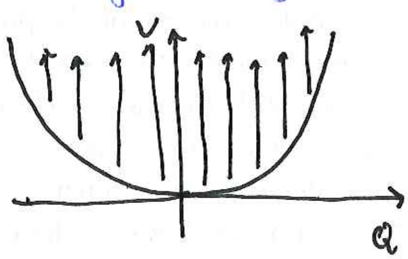
$$= \frac{-\alpha_{f.s}}{16M\pi^3} \int_R d\nu \int_0^{\infty} 4\pi q^2 \, dq \, \frac{T^{\mu}_{\mu}(i\nu, -q^2)}{-\nu^2 - \vec{q}^{\,2}}$$

$$Q^2 = \nu^2 + \vec{q}^{\,2}$$

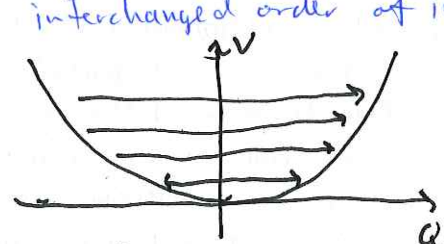
$$2Q\,dQ = 2q\,dq$$

$$= \frac{+\alpha_{f.s.}}{8M\pi^2} \int_{-\infty}^{\infty} d\nu \int_{\nu}^{\infty} 2Q\,dQ \, \frac{\sqrt{Q^2-\nu^2}}{Q^2} \, \underline{T^{\mu}_{\mu}(i\nu, Q^2)}$$

The integration region looks like



We can see the same region is covered by the interchanged order of integration



$$\int_{-\infty}^{\infty} d\nu \int_{\nu^2}^{\infty} dQ^2 \longrightarrow \int_0^{\infty} dQ^2 \int_{-Q}^{Q} d\nu$$

$$\Rightarrow \quad \delta M^{\gamma} = \frac{\alpha_{q.s.}}{8M\pi^2} \int_{0,R}^{\infty} dQ^2 \int_{-Q}^{Q} d\nu \, \frac{\sqrt{Q^2-\nu^2}}{Q^2} \, T^{\mu}_{\mu}(i\nu, Q^2)$$

b) $$T_{\mu\nu}(q^0, -q^2) = \frac{i}{2}\sum_{\sigma}\int d^4\xi \; e^{iq\cdot\xi} \langle p,\sigma | T\{J_\mu(\xi) J_\nu(0)\} | p\sigma\rangle \qquad (5)$$

Show the amplitude is crossing symmetric

$$T_{\nu\mu}(-q^0, -q^2) = T_{\mu\nu}(q^0, -q^2)$$

$$T_{\nu\mu}(-q^0, -q^2) = \frac{i}{2}\sum_{\sigma}\int d\xi^0 d^3\xi \; e^{-iq^0\xi^0 - i\vec{q}\cdot\vec{\xi}} \langle p,\sigma | T\{J_\nu(\xi) J_\mu(0)\} | p\sigma\rangle$$

– change dummy integration variable

$$x_i = -\xi_i \qquad \int_{-\infty}^{\infty} dx_i = \int_{-\infty}^{\infty} d\xi_i$$

$$T_{\nu\mu}(-q^0, -q^2) = \frac{i}{2}\sum_{\sigma}\int d\xi^0 d^3\xi \; e^{-iq^0\xi^0 + i\vec{q}\cdot\vec{\xi}} \langle p\sigma | T\{J_\nu(\xi^0, -\vec{\xi}) J_\mu(0)\} | p\sigma\rangle$$

The vector current is parity even

$$= \frac{i}{2}\sum_{\sigma}\int d\xi^0 d^3\xi \; e^{-iq^0\xi^0 + i\vec{q}\cdot\vec{\xi}} \langle p\sigma | T\{J_\nu(\xi) J_\mu(0)\} | p\sigma\rangle$$

Now we can use translation invariance to change

$$\langle p\sigma | T\{J_\nu(\xi) J_\mu(0)\} | p\sigma\rangle = \langle p\sigma | T\{J_\nu(0) J_\mu(-\xi)\} | p\sigma\rangle$$

Consider the first term in the time ordered expression, and use the translation operator

$$J_\mu(\xi) = e^{i\hat{p}\cdot\xi} J_\mu(0) e^{-i\hat{p}\cdot\xi}$$

$$\langle p\sigma| J_\nu(\xi) J_\mu(0) |p\sigma\rangle = \langle p\sigma| e^{i\hat{p}\cdot\xi} J_\nu(0) \underbrace{e^{-i\hat{p}\cdot\xi} e^{i\hat{p}\cdot\xi}}_{1} J_\mu(-\xi) e^{-i\hat{p}\cdot\xi} |p\sigma\rangle$$

$$= e^{ip\cdot\xi} \langle p\sigma| J_\nu(0) J_\mu(-\xi) |p\sigma\rangle e^{-ip\cdot\xi}$$

$$= \langle p\sigma| J_\nu(0) J_\mu(-\xi) |p\sigma\rangle$$

$$\Rightarrow T_{\nu\mu}(-q^0, -q^2) = \frac{i}{2} \sum_\sigma \int d^4\xi \, e^{-iq\cdot\xi} \Big[\theta(\xi^0) \langle p\sigma| J_\nu(0) J_\mu(-\xi) |p\sigma\rangle + \theta(-\xi^0) \langle p\sigma| J_\mu(-\xi) J_\nu(\xi) |p\sigma\rangle \Big]$$

Now change the entire dummy integration variable

$$\xi \to -\xi \qquad (\theta(-\xi^0) \to \theta(\xi^0))$$

$$\Rightarrow T_{\nu\mu}(-q^0, -q^2) = \frac{i}{2} \sum_\sigma \int d^4\xi \, e^{+iq\cdot\xi} \Big[\theta(-\xi^0) \langle p\sigma| J_\nu(0) J_\mu(\xi) |p\sigma\rangle + \theta(\xi^0) \langle p\sigma| J_\mu(\xi) J_\nu(0) |p\sigma\rangle \Big]$$

$$= T_{\mu\nu}(q^0, -q^2)$$

c) given the crossing symmetry of (5)

it is straightforward to see that (8)

$$T_{\mu\nu} = -\left(g_{\mu\nu} - q_\mu q_\nu/q^2\right) T_1(q^0, -q^2) + \frac{1}{M^2}\left(P_\mu - q_\mu \frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu \frac{P\cdot q}{q^2}\right) T_2(q^0, -q^2)$$

can only be crossing symmetric if

$$T_i(-q^0, -q^2) = T_i(q^0, -q^2)$$

This question was added after the hw was initially assigned, and not everyone realized this. So, this question will not be graded.

d) leading order QED amplitude of $e^-\gamma$ = ?

The spin averaged forward amplitude comes from 2 diagrams

$$\Rightarrow \quad T_{\mu\nu} = -\frac{1}{2}\sum_{\sigma} \bar{u}(p,\sigma)\left\{ \frac{\gamma_\mu(\not{p}+\not{q}+m)\gamma_\nu}{(p+q)^2-m^2+i\varepsilon} + \frac{\gamma_\nu(\not{p}-\not{q}+m)\gamma_\mu}{(p-q)^2-m^2+i\varepsilon}\right\} u(p,\sigma)$$

two insertions of $(-ie)$

- I have factored off the e^2 to keep the conventions of (4a)
- the spin average turns the expression into traces

$$T_{\mu\nu} = -\frac{1}{2}\left[\frac{\langle \bar{u}\gamma_\mu(\not{p}+\not{q}+m)\gamma_\nu u\rangle}{(p+q)^2-m^2+i\varepsilon} + \frac{\langle \bar{u}\gamma_\nu(\not{p}-\not{q}+m)\gamma_\mu u\rangle}{(p-q)^2-m^2+i\varepsilon}\right]$$

$$= -\frac{1}{2}\left[\frac{\langle (\not{p}+m)\gamma_\mu(\not{p}+\not{q}+m)\gamma_\nu\rangle}{(p+q)^2-m^2+i\varepsilon} + \frac{\langle (\not{p}+m)\gamma_\nu(\not{p}-\not{q}+m)\gamma_\mu\rangle}{(p-q)^2-m^2+i\varepsilon}\right]$$

To determine T_i (i) or t_i (ii) we need to perform the spin traces.

$$\langle (\not{p}+m)\gamma_\mu(\not{p}+\not{q}+m)\gamma_\nu\rangle = \langle \not{p}\gamma_\mu\not{p}\gamma_\nu\rangle + \langle \not{p}\gamma_\mu\not{q}\gamma_\nu\rangle + m^2\langle\gamma_\mu\gamma_\nu\rangle$$

$$= 8\,p_\mu p_\nu - 4\,p\cdot q\, g_{\mu\nu} + 4(p_\mu q_\nu + p_\nu q_\mu)$$

$$\langle (\not{p}+m)\gamma_\nu(\not{p}-\not{q}+m)\gamma_\nu\rangle = 8\,p_\mu p_\nu + 4\,p\cdot q\, g_{\mu\nu} - 4(p_\mu q_\nu + p_\nu q_\mu)$$

We can try to put the numerator structure in terms of the two Lorentz tensors.

1) $$T_{\mu\nu} = \underbrace{-\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)}_{D^{(1)}_{\mu\nu}} T_1 + \underbrace{\frac{1}{M^2}\left(P_\mu - q_\mu \frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu \frac{P\cdot q}{q^2}\right)}_{D^{(2)}_{\mu\nu}} T_2$$

$$\langle (\not{p}+m)\gamma_\mu(\not{p}+\not{q}+m)\gamma_\nu\rangle = 8P_\mu P_\nu - 4p\cdot q\, g_{\mu\nu} + 4(P_\mu q_\nu + P_\nu q_\mu)$$

$$= -D^{(1)}_{\mu\nu}\cdot 4p\cdot q + 8M^2 D^{(2)}_{\mu\nu} + \frac{8M^2}{M^2}\left(\frac{2p\cdot q + q^2}{2q^2}\right)\left[P_\mu q_\nu + P_\nu q_\mu - q_\mu q_\nu \frac{P\cdot q}{q^2}\right]$$

similarly, we find

$$\langle (\not{p}+m)\gamma_\nu(\not{p}-\not{q}+m)\gamma_\mu\rangle = -D^{(1)}_{\mu\nu}(-4p\cdot q) + 8M^2 D^{(2)}_{\mu\nu} + 8\left(\frac{p\cdot q}{q^2} - \frac{1}{2}\right)\left[P_\mu q_\nu + P_\nu q_\mu - q_\mu q_\nu \frac{P\cdot q}{q^2}\right]$$

We can combine these two terms by combining the denominators

$$\left[(p+q)^2 - m^2 + i\varepsilon\right]\left[(p-q)^2 - m^2 + i\varepsilon\right], \qquad p\cdot q = M\nu$$

$$= \left[q^2 + i\varepsilon + 2M\nu\right]\left[q^2 + i\varepsilon - 2M\nu\right]$$

$$= (q^2 + i\varepsilon)^2 - 4M^2\nu^2$$

$$= (-q^2 - i\varepsilon)^2 - 4M^2\nu^2 = (Q^2 - i\varepsilon)^2 - 4M^2\nu^2 \qquad (Q^2 = -q^2)$$

We have to combine like terms w/ $\pm$ signs, so we need

$$(p-q)^2 - m^2 \pm \left[(p+q)^2 - m^2\right]$$

$$= -2M\nu + q^2 \pm \left[2M\nu + q^2\right]$$

$$= -(Q^2 + 2M\nu) \mp (Q^2 - 2M\nu)$$

Returning to the amplitude, we have

$$T_{\mu\nu} = \frac{2M\nu\, D^{(1)}_{\mu\nu} - 4M^2 D^{(2)}_{\mu\nu} - 4\left(\frac{M\nu}{q^2}+\frac{1}{2}\right)\left[P_\mu q_\nu + P_\nu q_\mu - q_\mu q_\nu \frac{M\nu}{q^2}\right]}{(p+q)^2 - m^2 + i\varepsilon}$$

$$+ \frac{-2M\nu\, D^{(1)}_{\mu\nu} - 4M^2 D^{(2)}_{\mu\nu} - 4\left(\frac{M\nu}{q^2}-\frac{1}{2}\right)\left[P_\mu q_\nu + P_\nu q_\mu - q_\mu q_\nu \frac{M\nu}{q^2}\right]}{(p-q)^2 - m^2 + i\varepsilon}$$

$$= -D^{(1)}_{\mu\nu} \frac{8M^2\nu^2}{(Q^2-i\varepsilon)^2 - 4M^2\nu^2} + D^{(2)}_{\mu\nu} \frac{8M^2 Q^2}{(Q^2-i\varepsilon)^2 - 4M^2\nu^2}$$

$$\boxed{\begin{aligned} T_1(\nu,Q^2) &= \frac{8M^2\nu^2}{(Q^2-i\varepsilon)^2 - 4M^2\nu^2} = \frac{Q^4}{(Q^2-i\varepsilon)^2 - 4M^2\nu^2} - 1 \\ T_2(\nu,Q^2) &= \frac{8M^2 Q^2}{(Q^2-i\varepsilon)^2 - 4M^2\nu^2} \end{aligned}}$$

ii) We could repeat the exercise for the second parameterization, or just solve for t_i as functions of T_i

$$\boxed{\begin{aligned} t_1(\nu,Q^2) &= \frac{1}{Q^2}\left(T_1(\nu,Q^2) - \frac{\nu^2}{Q^2} T_2(\nu,Q^2)\right) \\ &= 0! \\ t_2(\nu,Q^2) &= \frac{1}{Q^2} T_2(\nu,Q^2) = \frac{8M^2}{(Q^2-i\varepsilon)^2 - 4M^2\nu^2} \end{aligned}}$$

e) Now we want to use this to evaluate the e^- self energy.

$$\delta M^\gamma = \lim_{Q_{uv}\to\infty} \left[\frac{\alpha_{f.s.}}{8M\pi^2} \int_0^{Q_{uv}^2} dQ^2 \int_{-Q}^{Q} d\nu \sqrt{Q^2-\nu^2}\, T_\mu^{\ \mu}(i\nu, Q^2) \left[\frac{1}{Q^2} - \frac{1}{Q^2+\Lambda^2} \right] + \delta M(\Lambda) \right]$$

The "little-t" parameterization will make our life easiear, as $t_1 = 0$. If we were to use a dispersion relation to determine T_1, we the trouble that would arise as

$$T_1(\nu, Q^2) = \frac{Q^4}{(Q^2-i\varepsilon)^2 - 4M^2\nu^2} - 1$$

there is a term without an imaginary piece (the "-1"), so an unsubtracted dispersion relation would miss this piece, if the infinite contour were ignored. However, since we have the complete expression, we do not need a dispersion integral, we can simpleg evaluate

$$T_\mu^{\ \mu}(i\nu, Q^2) = -3Q^2 t_1(i\nu, Q^2) + \left(1 + 2\frac{\nu^2}{Q^2}\right) Q^2 t_2(i\nu, Q^2)$$

$$= \left(1 + 2\frac{\nu^2}{Q^2}\right) Q^2 t_2(i\nu, Q^2)$$

$$= \left(1 + 2\frac{\nu^2}{Q^2}\right) \frac{8M^2Q^2}{(Q^2-i\varepsilon)^2 + 4M^2\nu^2}$$

note the sign from $(i\nu)^2$

$$\int_{-Q}^{Q} d\nu \sqrt{Q^2-\nu^2}\, T_\mu^{\ \mu}(i\nu, Q^2) = 2\pi MQ \left[4\left(\frac{Q^2}{4M^2}\right)^{3/2} + 2\sqrt{1+\frac{Q^2}{4M^2}} \left(1 - 2\frac{Q^2}{4M^2}\right) \right]$$

i)

$$\delta M^{\gamma} = \lim_{Q_{uv}\to\infty} \left[\frac{\alpha_{f.s.}}{4\pi} \int_0^{Q_{uv}^2} dQ^2\, Q\left[\frac{1}{Q^2} - \frac{1}{Q^2+\Lambda^2}\right]\left[4\tau^{3/2} + 2\sqrt{1+\tau}\,(1-2\tau)\right] + \delta M(\Lambda)\right]$$

$$\tau \equiv \frac{Q^2}{4M^2}$$

$$= \lim_{Q_{uv}\to\infty} \left[\frac{M\alpha_{f.s.}}{4\pi} \int_0^{Q_{uv}^2} dQ^2\, 2\sqrt{\frac{Q^2}{4M^2}}\left[\frac{1}{Q^2} - \frac{1}{Q^2+\Lambda^2}\right]\right]\left[4\tau^{3/2} + 2\sqrt{1+\tau}\,(1-2\tau)\right] + \delta M(\Lambda)\Big]$$

The first term $\left(\frac{1}{Q^2}\right)$ integrates to (in the large Q_{uv} limit)

$$\frac{M\alpha_{f.s.}}{4\pi}\left[\frac{3}{2} + 3\ln\left(\frac{Q_{uv}^2}{M^2}\right) + O\left(\frac{M^2}{Q_{uv}^2}\right)\right]$$

The second term $\left(\frac{1}{Q^2+\Lambda^2}\right)$ integrates to (in the large Q_{uv} limit)

$$\frac{M\alpha_{f.s.}}{4\pi}\left[3\ln\frac{Q_{uv}^2}{\Lambda^2} + O\left(\frac{1}{\Lambda^2}\right) + O\left(\frac{1}{Q_{uv}^2}\right)\right]$$

The sum of the two terms yields

$$\delta M^{\gamma} = \lim_{Q_{uv}\to\infty}\left[\frac{M\alpha_{f.s.}}{4\pi}\left(\frac{3}{2} + 3\ln\frac{Q_{uv}^2}{M^2} - 3\ln\frac{Q_{uv}^2}{\Lambda^2} + O\left(\frac{1}{\Lambda^2}\right) + O\left(\frac{1}{Q_{uv}^2}\right)\right) + \delta M(\Lambda)\right]$$

The limit is safe to take, resulting in

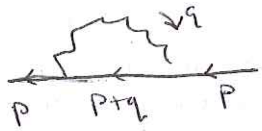
$$\delta M^{\gamma} = \frac{M\alpha}{4\pi}\left[\frac{3}{2} + 3\ln\frac{\Lambda^2}{M^2}\right] + \delta M(\Lambda)$$

ii) $$\frac{\partial}{\partial \ln \Lambda^2} \delta M^{\gamma} = 0 = \frac{M\alpha}{4\pi} \cdot 3 + \frac{\partial}{\partial \ln \Lambda^2} \delta M(\Lambda)$$

$$\Rightarrow \delta M(\Lambda) = \frac{M\alpha}{4\pi} \cdot 3 \ln\left(\frac{\Lambda_0^2}{\Lambda^2}\right)$$

iii) In dim-reg, the counter term contains a $1/\varepsilon$ pole as well as other terms. After $\overline{MS}$ renormalization (subtract $\frac{1}{\varepsilon} - \gamma + \ln 4\pi$), the coefficient of the log must be the same. This is because the log arises from IR physics: you can not approximate $\ln(m^2)$ with a finite number of local counter terms. So the coefficient of the $\ln(m^2)$ is universal between different renormalization schemes. However, the finite terms can be different.

e^- self-energy w/ dim-reg



$$iA = \mu^{4-d}\int\frac{d^dq}{(2\pi)^d}\,\bar{u}(p)(-ie\gamma^\mu)\,\frac{i}{\not{p}+\not{q}-m+i\varepsilon}\,(-ie\gamma^\nu)\,\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\,u(p)$$

$$= (i)^4(-)^3 e^2\mu^{4-d}\int\frac{d^dq}{(2\pi)^d}\,\frac{\bar{u}(p)\,\gamma^\mu(\not{p}+\not{q}+m)\gamma_\mu\,u(p)}{\cancel{\not{p}+\not{q}}\cdot[(p+q)^2-m^2+i\varepsilon][q^2+i\varepsilon]}$$

$$-i\bar{u}(p)\,\delta\Sigma\,u(p) = -e^2\bar{u}(p)\,\mu^{4-d}\int\frac{d^dq}{(2\pi)^d}\,\frac{\gamma^\mu(\not{p}+\not{q}+m)\gamma_\mu}{[(p+q)^2-m^2+i\varepsilon][q^2+i\varepsilon]}\;u(p)$$

$$iS = i[\not{p}-m-\delta\Sigma(p)]^{-1}$$

$$\Rightarrow \delta\Sigma = -ie^2\mu^{4-d}\int\frac{d^dq}{(2\pi)^d}\left[-(d-2)(\not{p}+\not{q})+md\right]\int_0^1 dx\,\frac{1}{[q^2+2xp\cdot q+\underset{+xp^2}{\cancel{xq^2}}-xm^2+i\varepsilon]^2}$$

$$= -ie^2\mu^{4-d}\int\frac{d^dq}{(2\pi)^d}\int_0^1 dx\,\frac{2m-(d-2)\not{q}}{[(q+xp)^2-x^2p^2+\underset{+xp^2}{\cancel{xq^2}}-xm^2+i\varepsilon]^2}$$

$$= -ie^2\int_0^1 dx\,\mu^{4-d}\int\frac{d^dl}{(2\pi)^d}\,\frac{2m-(d-2)(\not{l}-x\not{p})}{[l^2-m^2x(1+x)+\underset{+xp^2}{\cancel{xq^2}}+i\varepsilon]^2}$$

$$= -ie^2\int_0^1 dx\,\mu^{4-d}\int\frac{d^dl}{(2\pi)^d}\,\frac{m(2+x(d-2))}{[l^2-\Delta]^2}\,,\quad \Delta = m^2x(1+x)-\cancel{xq^2}+xp^2$$
$$= m^2x^2+x(m^2-p^2)$$

$$= -ie^2\int_0^1 dx\,m(2(1-x)+xd)\mu^{4-d}\,\frac{i}{(4\pi)^{d/2}}\,\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\,,\qquad d=4-2\varepsilon,\quad \frac{d}{2}=2-\varepsilon,\quad 2-\frac{d}{2}=\varepsilon$$

$$= \frac{me^2}{(4\pi)^2}\int_0^1 dx\,[2(1-x)+xd]\,\mu^{2\varepsilon}(4\pi)^\varepsilon\,\Gamma(\varepsilon)\,\frac{1}{\Delta^\varepsilon}$$

$$\delta\Sigma = \frac{m\alpha}{4\pi}\int_0^1 dx\,\left[2(1-x)+xd\right]\left[\frac{1}{\varepsilon}-\gamma+\ln 4\pi+\ln\frac{\mu^2}{m^2}-\ln x^2\right]$$

$d = 4-2\varepsilon$

for self-energy we can focus on on-shell $p^2 = m^2$

$$= \frac{m\alpha}{4\pi}\cdot\int_0^1 dx\left[2(1+x)\left[\frac{1}{\varepsilon}-\gamma+\ln 4\pi+\ln\frac{\mu^2}{m^2}-\ln x^2\right]-2x\right]$$

$$= \frac{m\alpha}{4\pi}\cdot 2\left\{\frac{3}{2}\left[\frac{1}{\varepsilon}-\gamma+\ln 4\pi+\ln\frac{\mu^2}{m^2}\right]-\frac{1}{2}+\frac{5}{2}\right\}$$

$$= \frac{m\alpha}{4\pi}\cdot 3\left[\frac{1}{\varepsilon}-\gamma+\ln 4\pi+\ln\frac{\mu^2}{m^2}+\frac{4}{3}\right]$$

$$\boxed{\delta\bar{\Sigma} = \frac{m\alpha}{4\pi}\left[4+3\ln\frac{\mu^2}{m^2}\right]} \Rightarrow \delta M(\mu) = 3\frac{m\alpha}{4\pi}\ln\frac{\mu_0^2}{\mu^2}$$

Comparing with Pauli-Villars, we see

$$\boxed{\delta M^{dr}(\mu) = \delta M^{PV}(\mu) - \frac{5}{2}}$$